

The variety of three-dimensional real Jordan algebras

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Abstract

In this paper, we study the variety \mathcal{Jor}_3 of three-dimensional Jordan algebras over the field of real numbers. We establish the list of 26 non-isomorphic Jordan algebras and describe the irreducible components of \mathcal{Jor}_3 proving that it is the union of Zariski closure of the orbits of 8 rigid algebras.

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1 Introduction

Let V be an n -dimensional vector space over a field \mathbf{k} of characteristic $\neq 2$, then the bilinear maps $V \times V \rightarrow V$ form a vector space $\text{Hom}_{\mathbf{k}}(V \times V, V) = V^* \otimes V^* \otimes V$ of dimension n^3 which has structure of an affine variety \mathbf{k}^{n^3} . The algebras satisfying the Jordan identity form a Zariski-closed affine subset of \mathbf{k}^{n^3} which we call the variety of Jordan \mathbf{k} -algebras of dimension n and denote it by \mathcal{Jor}_n . Each point of $\mathcal{Jor}_n \subseteq \mathbf{k}^{n^3}$ will be seen as an n^3 -tuple of structure constants (c_{ij}^k) with $i, j, k \in \{1, 2, 3\}$ and it represent a Jordan \mathbf{k} -algebra of dimension n , with respect to some fixed basis.

The linear general group $G = \text{GL}(V)$ operates on \mathcal{Jor}_n by conjugation, decomposing the variety into G -orbits which correspond to the classes of isomorphic Jordan algebras. If an algebra \mathcal{J} lies in the Zariski closure of the orbit of a (non-isomorphic) algebra \mathcal{J}' in the variety, then we will say that \mathcal{J}' is a deformation of \mathcal{J} . An algebra \mathcal{J} whose G -orbit, \mathcal{J}^G , is Zariski-open in \mathcal{Jor}_n is called rigid.

The goal of this paper is to classify algebraically and geometrically Jordan algebras of dimension three over the field \mathbb{R} of real numbers. By geometric classification we mean the problem of determine the orbits of \mathcal{Jor}_n under the action of G , find all possible deformation between the algebras and, finally, determine the complete list of rigid algebras since the closure of the orbits of such algebras generates an irreducible component of the variety.

In this work we study the variety $\mathcal{Jor}_3 \subseteq \mathbb{R}^{27}$ of real Jordan algebras of dimension three. Firstly, we work in the algebraic classification up to isomorphism of these algebras. This allowed us to determine that the number of G -orbits in the variety \mathcal{Jor}_3 correspond to 26. Next, given a three-dimensional real Jordan algebra we determined if it is rigid or we find a deformation of its to finally conclude that the variety \mathcal{Jor}_3 has 8 irreducible components.

The classification problem of the algebraic structures of given dimension has been extensively studied. The only class of algebras (associative, Jordan and Lie) which is completely described in any dimension is the one of semi-simple algebras. In general, the list of all algebras is known only for small dimensions.

The literature, when the base field is algebraically closed is extensive. Namely, in 1975, P. Gabriel presented in [4] the lists of all unitary associative algebras up to dimension four and described all rigid algebras of such variety. Later, G. Mazzola, in his work [13], extended the algebraic and geometric classification to dimension five and he proved that there are ten irreducible components in this variety. In the context of Lie algebras both classification are known for dimension till six, see [10]. As to Jordan algebras, H. Wesseler in [16] described unitary Jordan algebras up to dimension six and M. E. Martin in [12] classified algebraically all Jordan algebras (associative and non-associative, unitary and non-unitary) up to dimension four. Regarding the description of the variety \mathcal{Jor}_n the references are rather recent. In [9, 2005] I. Kashuba and I. Shestakov described the irreducible components of \mathcal{Jor}_3 and in [7, 2006] the first author classified geometrically the unitary Jordan algebras of dimensions four and five. In [2, 2011], the authors determined the laws and deformations of nilpotent Jordan algebras of dimension three and four over the field of complex numbers. Finally, in [8, 2014] I. Kashuba and M. E. Martin generalized these results to obtain a complete description for the varieties \mathcal{Jor}_n for $n \leq 4$.

When we consider the field of real numbers, the results in the literature for Jordan algebras (and even Lie and associative algebras) are scarce, including for small dimensions. In 2007, Ancochea Bermúdez and others, in their work [3], classified algebraically and

³Note that analogously one defines the varieties of associative and Lie algebras.

geometrically associative algebras of dimension two, and afterwards, in [1], the authors did an analogous study related to two-dimensional real Jordan algebras.

The paper is organized as follows. In Section 2, we recall the basic concepts and necessary results for finite-dimensional Jordan algebras. Also, we present the methods used to study the irreducible components of the variety of Jordan algebras and the deformations between these algebras. In particular, we show that the dimension of the group of 2-cocycles of an algebra does not increase under deformations. In Section 3, we classify all real Jordan algebras of dimension three and also show that all algebras are pairwise non-isomorphic. Finally, in Section 4 we construct deformations between algebras in \mathcal{Jor}_3 and describe its irreducible components.

2 Preliminaries

In this section we start with the basic concepts, notations and principal results about finite-dimensional Jordan algebras over a field \mathbf{k} of characteristic 0, and in the second part we will introduce the variety of n -dimensional Jordan algebras, \mathcal{Jor}_n , together with its properties.

For the standard terminology on Jordan algebras, the reader is referred to the book of N. Jacobson [6], for concepts from deformation theory see [5].

Definition 1. A **Jordan \mathbf{k} -algebra** is a commutative algebra \mathcal{J} with a multiplication " \cdot " satisfying the Jordan identity:

$$((x \cdot x) \cdot y) \cdot x = (x \cdot x) \cdot (y \cdot x), \quad \text{for any } x, y \in \mathcal{J} \quad (2)$$

or, equivalently, its linearization

$$(x, y, z \cdot w) + (w, y, z \cdot x) + (z, y, x \cdot w) = 0, \quad (3)$$

for any $x, y, z, w \in \mathcal{J}$. Here $(x, y, z) := (x \cdot y) \cdot z - x \cdot (y \cdot z)$ is the associator of x, y, z .

Example 4. Let (\mathcal{U}, j) be an associative algebra with an involution j . Then

$$H(\mathcal{U}, j) = \{u \in \mathcal{U} \mid u = j(u)\},$$

the set of elements symmetric with respect to j together with the multiplication given by the formula $x \odot y = \frac{1}{2}(x \cdot y + y \cdot x)$, where \cdot is the multiplication in \mathcal{U} , is a Jordan algebra.

Example 5. Let V be a vector space over \mathbf{k} with a symmetric bilinear form $f = f(x, y)$ on V . Then the set $\mathcal{J}(V, f) = \mathbf{k}1 + V$ endowed with the multiplication

$$(\alpha 1 + x)(\beta 1 + y) = (\alpha\beta + f(x, y))1 + \alpha y + \beta x \quad \text{for } \alpha, \beta \in \mathbf{k} \quad x, y \in V$$

forms a Jordan algebra called the *Jordan algebra of the symmetric bilinear form f* .

For any Jordan algebra \mathcal{J} we define inductively a series of subsets by setting

$$\begin{aligned} \mathcal{J}^1 &= \mathcal{J}^{(1)} = \mathcal{J}, \\ \mathcal{J}^n &= \mathcal{J}^{n-1} \cdot \mathcal{J} + \mathcal{J}^{n-2} \cdot \mathcal{J}^2 + \dots + \mathcal{J} \cdot \mathcal{J}^{n-1}, \\ \mathcal{J}^{(n)} &= \mathcal{J}^{(n-1)} \cdot \mathcal{J}. \end{aligned}$$

For any $i \geq 1$, both \mathcal{J}^i and $\mathcal{J}^{(i)}$ are ideals of the algebra \mathcal{J} . The chain $\mathcal{J}^{(1)} \supseteq \mathcal{J}^{(2)} \supseteq \dots \supseteq \mathcal{J}^{(n)} \supseteq \dots$ is called the **lower central series of \mathcal{J}** . The subset \mathcal{J}^n is called the **n -th power of the algebra \mathcal{J}** . Observe that $\mathcal{J}^i = \mathcal{J}^{(i)}$, for $i = 1, 2, 3$.

Definition 6. A Jordan algebra \mathcal{J} is called **nilpotent** if there exists an integer s such that $\mathcal{J}^{(s)} = 0$. The minimal integer for which this condition holds is the **nilindex** of \mathcal{J} .

For a nilpotent algebra \mathcal{J} of nilindex s define the **nilpotency type** of \mathcal{J} as the sequence $(n_1, n_2, n_3, \dots, n_{s-1})$, where $n_i = \dim(\mathcal{J}^{(i)} / \mathcal{J}^{(i+1)})$. Observe that all $n_i > 0$.

Definition 7. A Jordan algebra \mathcal{J} is called:

1. **simple** if 0 and \mathcal{J} are the only ideals of \mathcal{J} and $\mathcal{J}^2 \neq 0$.
2. **semi-simple** if it is a direct sum of simple algebras.
3. **central simple** if $\mathcal{J}_K = \mathcal{J} \otimes_{\mathbf{k}} K$ is simple for any extension K of \mathbf{k} .

The following theorem classifies all finite-dimensional central simple Jordan algebras.

Theorem 8. [6, V.7] Let \mathcal{J} be a finite-dimensional central simple Jordan algebra over \mathbf{k} . Then we have the following possibilities for \mathcal{J} :

- i. $\mathcal{J} = \mathbf{k}$,
- ii. $\mathcal{J} = \mathcal{J}(V, f)$, the Jordan algebra of a non-degenerate symmetric bilinear form f on a finite-dimensional \mathbf{k} -vector space V such that $\dim V > 1$,
- iii. $\mathcal{J} = H(\mathcal{U}, j)$, where (\mathcal{U}, j) is a finite-dimensional central simple associative algebra with involution j of degree $n \geq 3$, or
- iv. \mathcal{J} is an algebra such that there exists a finite extension field K of the base field \mathbf{k} such that $\mathcal{J}_K \simeq H(M_3(\mathfrak{C}_K), \tau)$ where $M_3(\mathfrak{C}_K)$ is the algebra of all 3×3 matrices with elements in a Cayley algebra \mathfrak{C} over K and τ is the standard involution conjugate transpose.

The following proposition is known as the Wedderburn Principal Theorem, in this case is formulated for finite-dimensional Jordan algebras over a field of characteristic 0.

Proposition 9. [14, p. 405] Let \mathcal{J} be a finite-dimensional Jordan algebra over a field \mathbf{k} of characteristic 0, and let $N = \text{Rad}(\mathcal{J})$ be the radical of \mathcal{J} (i.e. the unique maximal nilpotent ideal of \mathcal{J}). Then there exists a subalgebra \mathcal{J}_{ss} of \mathcal{J} such that $\mathcal{J} = \mathcal{J}_{ss} \oplus N$ (as vector spaces) and $\mathcal{J}_{ss} \simeq \mathcal{J}/N$.

Moreover, the quotient $\mathcal{J}_{ss} := \mathcal{J}/N$ is semi-simple, has an identity element and its decomposition into simple components is unique, see [6, V.2, V.5]. The identity element \bar{e} of \mathcal{J}_{ss} could be lifted to an idempotent e of \mathcal{J} . Then, without loss of generality, we may assume that all finite-dimensional Jordan algebra over a field \mathbf{k} of $\text{char } \mathbf{k} = 0$ either is nilpotent or has an idempotent element. Thus we have:

Theorem 10. [6, III.1] Let e be an idempotent in \mathcal{J} then we have the following decomposition into a direct sum of subspaces

$$\mathcal{J} = \mathcal{P}_1 \oplus \mathcal{P}_{\frac{1}{2}} \oplus \mathcal{P}_0,$$

where $\mathcal{P}_i = \{x \in \mathcal{J} \mid x \cdot e = ix\}$, for $i = 0, \frac{1}{2}, 1$.

This decomposition is called the **Peirce decomposition of \mathcal{J} relative to idempotent e** . The multiplication table for the Peirce components \mathcal{P}_i is:

$$\begin{aligned} \mathcal{P}_1^2 &\subseteq \mathcal{P}_1, & \mathcal{P}_1 \cdot \mathcal{P}_0 &= 0, & \mathcal{P}_0^2 &\subseteq \mathcal{P}_0, \\ \mathcal{P}_0 \cdot \mathcal{P}_{\frac{1}{2}} &\subseteq \mathcal{P}_{\frac{1}{2}}, & \mathcal{P}_1 \cdot \mathcal{P}_{\frac{1}{2}} &\subseteq \mathcal{P}_{\frac{1}{2}}, & \mathcal{P}_{\frac{1}{2}}^2 &\subseteq \mathcal{P}_0 \oplus \mathcal{P}_1. \end{aligned} \quad (11)$$

Furthermore, we have the following generalization: if \mathcal{J} is a Jordan algebra with an identity element which is a sum of pairwise orthogonal idempotents e_i , i.e. $1 = \sum_{i=1}^n e_i$, we have the refined **Peirce decomposition of \mathcal{J} relative to idempotents $\{e_1, \dots, e_n\}$** :

$$\mathcal{J} = \bigoplus_{1 \leq i \leq j \leq n} \mathcal{P}_{ij} \quad (12)$$

where $\mathcal{P}_{ii} = \{x \in \mathcal{J} \mid x \cdot e_i = x\}$ and $\mathcal{P}_{ij} = \{x \in \mathcal{J} \mid x \cdot e_i = x \cdot e_j = \frac{1}{2}x\}$. The multiplication table for the Peirce components is:

$$\begin{aligned} \mathcal{P}_{ii}^2 &\subseteq \mathcal{P}_{ii}, & \mathcal{P}_{ij} \cdot \mathcal{P}_{ii} &\subseteq \mathcal{P}_{ij}, & \mathcal{P}_{ij}^2 &\subseteq \mathcal{P}_{ii} \oplus \mathcal{P}_{jj}, \\ \mathcal{P}_{ij} \cdot \mathcal{P}_{jk} &\subseteq \mathcal{P}_{ik}, & \mathcal{P}_{ii} \cdot \mathcal{P}_{jj} &= \mathcal{P}_{ii} \cdot \mathcal{P}_{jk} = \mathcal{P}_{ij} \cdot \mathcal{P}_{kl} = 0, \end{aligned} \quad (13)$$

where the indices i, j, k, l are all different. Note that the Peirce decomposition is inherited for ideals of \mathcal{J} .

Let V be an n -dimensional \mathbf{k} -vector space with a fixed basis $\{e_1, e_2, \dots, e_n\}$. To endow V with a Jordan \mathbf{k} -algebra structure, (\mathcal{J}, \cdot) , it suffices to specify n^3 structure constants $c_{ij}^k \in \mathbf{k}$, namely,

$$e_i \cdot e_j = \sum_{k=1}^n c_{ij}^k e_k, \quad i, j \in \{1, 2, \dots, n\}.$$

The choice of c_{ij}^k is not arbitrary, it must reflect the fact that Jordan algebras are commutative and satisfy Jordan identity (2). Therefore we obtain:

$$\begin{aligned} c_{ij}^k &= c_{ji}^k, \\ \sum_{a=1}^n c_{ij}^a \sum_{b=1}^n c_{kl}^b c_{ab}^p - \sum_{a=1}^n c_{kl}^a \sum_{b=1}^n c_{ja}^b c_{ib}^p + \sum_{a=1}^n c_{lj}^a \sum_{b=1}^n c_{ki}^b c_{ab}^p - \\ \sum_{a=1}^n c_{ki}^a \sum_{b=1}^n c_{ja}^b c_{lb}^p + \sum_{a=1}^n c_{kj}^a \sum_{b=1}^n c_{il}^b c_{ab}^p - \sum_{a=1}^n c_{il}^a \sum_{b=1}^n c_{ja}^b c_{kb}^p &= 0, \end{aligned} \quad (14)$$

for all $i, j, k, l, p \in \{1, 2, \dots, n\}$. Polynomial equations (14) cut out an **algebraic variety** $\mathcal{J}_{\text{or}_n}$ in $\mathbf{k}^{n^3} = V^* \otimes V^* \otimes V$. A point $(c_{ij}^k) \in \mathcal{J}_{\text{or}_n}$ represents an n -dimensional \mathbf{k} -algebra \mathcal{J} along with a particular choice of basis (which gives the structure constants c_{ij}^k). A change of basis in \mathcal{J} gives rise to a possible different point of $\mathcal{J}_{\text{or}_n}$ or, equivalently, the general linear group $G = \text{GL}(V)$ operates on $\mathcal{J}_{\text{or}_n}$ via “conjugation”:

$$g(\mathcal{J}, \cdot) \mapsto (\mathcal{J}, \cdot_g), \quad x \cdot_g y = g(g^{-1}x \cdot g^{-1}y), \quad (15)$$

for any $\mathcal{J} \in \mathcal{J}_{\text{or}_n}$, $g \in G$ and $x, y \in V$. The **G-orbit** of a Jordan algebra \mathcal{J} , that is, the set of all images of \mathcal{J} under the action of G , is denoted by \mathcal{J}^G . The set of different G -orbits of this action is in one-to-one correspondence with the set of isomorphism classes of n -dimensional Jordan algebras. Now, we can consider the inclusion diagrams of the Zariski closure of orbits of n -dimensional Jordan algebras. To relate the orbits, we say that \mathcal{J}_1 is a **deformation** of \mathcal{J}_2 or that \mathcal{J}_1 **dominates** \mathcal{J}_2 and denote this by $\mathcal{J}_1 \rightarrow \mathcal{J}_2$, if the orbit \mathcal{J}_2^G

is contained in the Zariski closure of the orbit \mathcal{J}_1^G . A Jordan algebra \mathcal{J} is called *rigid* if its G -orbit, \mathcal{J}^G , is a Zariski-open set in $\mathcal{J}_{\text{or}_n}$. In terms of deformation if \mathcal{J}_1 is a deformation of a rigid algebra \mathcal{J} then $\mathcal{J}_1^G \cap \mathcal{J}^G \neq \emptyset$ and therefore $\mathcal{J}_1 \simeq \mathcal{J}$.

The rigid algebras are of particular interest. Indeed, $\mathcal{J}_{\text{or}_n}$ as any affine variety could be decomposed into its irreducible components. Then if $\mathcal{J} \in \mathcal{J}_{\text{or}_n}$ is rigid, there exists an irreducible component T such that $\mathcal{J}^G \cap T$ is a non-empty open subspace in T , and therefore the closure of \mathcal{J}^G contains T .

The most known sufficient condition for an algebra to be rigid is given in terms of its cohomology group. We say that the second cohomology group $H^2(\mathcal{J}, \mathcal{J})$ of a Jordan algebra \mathcal{J} with coefficients in itself vanishes if for every bilinear mapping $h : \mathcal{J} \times \mathcal{J} \rightarrow \mathcal{J}$ satisfying

$$\begin{aligned} h(a, b) &= h(b, a) \\ (h(a, a)b)a + h(a^2, b)a + h(a^2b, a) &= a^2h(b, a) + h(a, a)(ba) + h(a^2, ba) \end{aligned} \quad (16)$$

for all $a, b \in \mathcal{J}$ there exists a linear mapping $\mu : \mathcal{J} \rightarrow \mathcal{J}$ such that

$$h(a, b) = \mu(ab) - a\mu(b) - \mu(a)b. \quad (17)$$

The bilinear mapping h is called **2-cocycle** of \mathcal{J} and we denote the set of all 2-cocycles of \mathcal{J} by $Z^2(\mathcal{J}, \mathcal{J})$. For the precise definition of these groups for Jordan algebras we refer to [6].

Proposition 18. *If the second cohomology group $H^2(\mathcal{J}, \mathcal{J})$ of a Jordan algebra \mathcal{J} with coefficients in itself vanishes, then \mathcal{J} is rigid. In particular it follows that any semi-simple Jordan algebra is rigid.*

It was originally obtained in [5] for associative and Lie algebras, for the case of Jordan algebras the proof is analogous, see [7].

Example 19. Consider $\mathcal{J}_{12} \in \mathcal{J}_{\text{or}_3}$ the Jordan algebra with basis $\{e_1, n_1, n_2\}$ and multiplication given by $e_1^2 = e_1$ and $e_1 \cdot n_i = \frac{1}{2}n_i$ for $i = 1, 2$. Let $h : \mathcal{J}_{12} \times \mathcal{J}_{12} \rightarrow \mathcal{J}_{12}$ be a bilinear map satisfying (16), then

$$h(e_1, e_1) = \alpha e_1, \quad h(e_1, n_i) = \beta_i e_1 + \frac{\alpha}{2}n_i, \quad h(n_i, n_j) = \beta_j n_i + \beta_i n_j$$

for any $\alpha, \beta_i \in \mathbb{R}$ and $1 \leq i, j \leq 2$. Define a linear mapping $\mu : \mathcal{J}_{12} \rightarrow \mathcal{J}_{12}$ as been $\mu(e_1) = -\alpha e_1$ and $\mu(n_i) = -2\beta_i e_1 + n_i$ for $i = 1, 2$, then (17) holds and thus $H^2(\mathcal{J}_{12}, \mathcal{J}_{12}) = 0$, which implies \mathcal{J}_{12} is rigid.

We will construct the deformations between Jordan algebras using the following property. Let $\{e_1, \dots, e_n\}$ be a basis of \mathcal{J} as a vector space. Then the multiplication in algebra \mathcal{J} is defined by n^3 structure constants c_{ij}^k . Let

$$g(t) \in \text{Mat}_n(\mathbf{k}[t]) \quad (20)$$

be a change of basis of \mathcal{J} such that for any $t \neq 0$ it is non-degenerate, i.e. $g(t) \in G$. We denote by \mathcal{J}_t the algebra obtained from \mathcal{J} by the change of basis $g(t)$ and let $c_{ij}^k(t)$ denote the corresponding structure constants. Note that by (15) $\mathcal{J}_t \simeq \mathcal{J}$ for any $t \neq 0$. Then if \mathcal{J}_1 is a Jordan algebra defined by structure constants $d_{ij}^k = c_{ij}^k(0)$ with respect to the same basis $\{e_1, \dots, e_n\}$ then $\mathcal{J}_1 \in \overline{\mathcal{J}^G}$ and thus \mathcal{J} is a deformation of \mathcal{J}_1 .

In the following proposition we collect properties which will be used to show that there is no deformation between certain algebras.

Proposition 21. *Let $\mathcal{J}, \mathcal{J}_1 \in \mathcal{Jor}_n$ and $\mathcal{J} \rightarrow \mathcal{J}_1$. Then*

- (i) $\dim \text{Aut}(\mathcal{J}) < \dim \text{Aut}(\mathcal{J}_1)$, where $\text{Aut}(\mathcal{J}) < G$ is the automorphism group of \mathcal{J} .
- (ii) $\dim \text{Rad}(\mathcal{J}) \leq \dim \text{Rad}(\mathcal{J}_1)$.
- (iii) $\dim \text{Ann}(\mathcal{J}) \leq \dim \text{Ann}(\mathcal{J}_1)$, where $\text{Ann}(\mathcal{J}) = \{a \in \mathcal{J} \mid a\mathcal{J} = 0\}$ is the annihilator of \mathcal{J} .
- (iv) $\dim \mathcal{J}^r \geq \dim \mathcal{J}_1^r$, for any positive integer r .
- (v) If also $\mathcal{J}', \mathcal{J}'_1 \in \mathcal{Jor}_{n'}$ and $\mathcal{J}' \rightarrow \mathcal{J}'_1$, then $\mathcal{J} \oplus \mathcal{J}' \rightarrow \mathcal{J}_1 \oplus \mathcal{J}'_1$.
- (vi) Any polynomial identity of \mathcal{J} is valid in \mathcal{J}_1 . In particular, any deformation of a non-associative algebra is again non-associative.
- (vii) $\dim Z^2(\mathcal{J}, \mathcal{J}) \leq \dim Z^2(\mathcal{J}_1, \mathcal{J}_1)$.

Proof. For the proof of (i) to (vi) we refer to [8]. To see (vii), let $\{e_1, e_2, \dots, e_n\}$ be a basis for \mathcal{J} and $h \in Z^2(\mathcal{J}, \mathcal{J})$. Then h is completely define by the n^3 constants $\alpha_{ij}^k \in \mathbb{R}$ given by $h(e_i, e_j) = \sum_{k=1}^n \alpha_{ij}^k e_k$.

Linearizing the identity of 2-cocycle (16) we have:

$$\begin{aligned} & (h(x, y)w)z + (h(x, z)w)y + (h(y, z)w)x + h((xy)w, z) + h((xz)w, y) + \\ & + h((yz)w, x) + h(xy, w)z + h(xz, w)y + h(yz, w)x = (xy)h(w, z) + \\ & + (xz)h(w, y) + (yz)h(w, x) + h(x, y)(wz) + h(x, z)(wy) + h(y, z)(wx) + \\ & + h(xy, wz) + h(xz, wy) + h(yz, wx). \end{aligned}$$

Computing this identity and the commutative condition of h in the basis, it results in $l(n) = \left(n^4 + \frac{n(n-1)}{2}\right)n$ equations having α_{ij}^k as unknowns. Then $\dim Z^2(\mathcal{J}, \mathcal{J}) = n^3 - \text{rank}(P_{l(n), n^3})$, where $P_{l(n), n^3}$ represent the matrix of the system of equations, thus $\dim Z^2(\mathcal{J}, \mathcal{J}) \geq s$ is equivalent to the fact that all $(n^3 - s + 1)$ -minors of $P_{l(n), n^3}$ vanish. Then the set $\{\mathcal{J} \in \mathcal{Jor}_n \mid \dim Z^2(\mathcal{J}, \mathcal{J}) \geq s\}$ is Zariski-closed. \square

3 Real Jordan algebras of small dimensions

In this section we present the lists of all one and two dimensional indecomposable Jordan algebras over \mathbb{R} . Further we describe all non-isomorphic, three-dimensional real Jordan algebras.

Recall that $\mathcal{J} = \mathcal{J}_{ss} \oplus N$, where N denotes the radical of \mathcal{J} and \mathcal{J}_{ss} is semi-simple. We will denote by e_i the elements in \mathcal{J}_{ss} and by n_i the ones which belong to N . Henceforth, for convenience we drop \cdot and denote the multiplication in \mathcal{J} simply as xy .

3.1. Real Jordan algebras of dimension one

There are two non-isomorphic one-dimensional real Jordan algebras: the simple algebra $\mathbb{R}e$, with $e^2 = e$ and the nilpotent algebra $\mathbb{R}n$, where $n^2 = 0$.

3.2. Real Jordan algebras of dimension two

In [1] all two dimensional real Jordan algebras are described. Using their list we have the following 4 indecomposable algebras:

\mathcal{B}	Multiplication Table	Observation
\mathcal{B}_1	$e_1^2 = e_1 \quad e_1 n_1 = n_1 \quad n_1^2 = 0$	associative
\mathcal{B}_2	$e_1^2 = e_1 \quad e_1 n_1 = \frac{1}{2}n_1 \quad n_1^2 = 0$	
\mathcal{B}_3	$n_1^2 = n_2 \quad n_1 n_2 = 0 \quad n_2^2 = 0$	associative, nilpotent
\mathcal{B}_4	$e_1^2 = e_1 \quad e_1 e_2 = e_2 \quad e_2^2 = -e_1$	associative, simple

Table 1: Indecomposable two-dimensional Jordan algebras over \mathbb{R} .

3.3. Real Jordan algebras of dimension three

The description of three-dimensional real Jordan algebras is organized according to the dimension of the radical and subsequently the possible values of the nilpotency type. Also for each algebra we calculate the dimensions of its automorphism group $\text{Aut}(\mathcal{J})$ and the annihilator $\text{Ann}(\mathcal{J})$.

We denote by $\mathcal{J}^\# = \mathcal{J} \oplus \mathbb{R}1$ the Jordan algebra obtained by formal adjoining of the identity element 1 of \mathbb{R} .

3.3.1. Semisimple Jordan algebras

Any simple Jordan algebra can be considered as a central simple algebra over its centroid which is a field, see [15, p.13]. As a consequence we can reduce the problem of classify finite dimensional simple Jordan algebras over \mathbb{R} to the problem of classify central simple ones over a finite extension of \mathbb{R} , i. e. \mathbb{C} . Thus, by Theorem 8 if \mathcal{J} is a simple Jordan algebra of dimension ≤ 3 over \mathbb{R} then:

- i) $\mathcal{J} = \mathbb{R}e$ of dimension one,
- ii) $\mathcal{J} = \mathcal{B}_4$ of dimension two (that is the field of the complex numbers),

And if V is a two-dimensional real vector space with basis $\{e_1, e_2\}$ then:

- iii) $\mathcal{J} = \mathcal{J}(V, f_1)$ of dimension three, where f_1 is the non-degenerate symmetric bilinear form: $f_1(e_1, e_1) = 1$, $f_1(e_2, e_2) = -1$ and $f_1(e_1, e_2) = 0$,
- iv) $\mathcal{J} = \mathcal{J}(V, f_2)$ of dimension three, where f_2 is given by: $f_2(e_i, e_i) = -1$ for $i = 1, 2$ and $f_2(e_1, e_2) = 0$,
- v) $\mathcal{J} = \mathcal{J}(V, f_3)$ of dimension three, where f_3 is given by: $f_3(e_i, e_i) = 1$ for $i = 1, 2$ and $f_3(e_1, e_2) = 0$.

Considering direct sum of these algebras we obtain all semi-simple real Jordan algebras of dimension three:

\mathcal{J}	Multiplication Table	$\dim \text{Aut}(\mathcal{J})$	$\dim \text{Ann}(\mathcal{J})$	Observation
\mathcal{J}_1	$\mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_3$	0	0	associative unitary
\mathcal{J}_2	$\mathcal{B}_4 \oplus \mathbb{R}e_3$	0	0	associative unitary
\mathcal{J}_3	$e_2^2 = e_1 \quad e_3^2 = -e_1$ $e_1 e_i = e_i \quad i = 1, 2, 3$	1	0	unitary $\mathcal{J}(V, f_1)$
\mathcal{J}_4	$e_2^2 = e_3^2 = -e_1 \quad e_1 e_i = e_i \quad i = 1, 2, 3$	1	0	unitary $\mathcal{J}(V, f_2)$

\mathcal{J}	Multiplication Table	$\dim \text{Aut}(\mathcal{J})$	$\dim \text{Ann}(\mathcal{J})$	Observation
\mathcal{J}_5	$e_2^2 = e_3^2 = e_1 \quad e_1 e_i = e_i \quad i = 1, 2, 3$	1	0	unitary $\mathcal{J}(\mathbb{V}, f_3)$

Table 2: Three-dimensional semi-simple Jordan algebras over \mathbb{R} .

3.3.2. Jordan algebras with one-dimensional radical

Consider \mathcal{J} a real Jordan algebra of dimension three with $\dim N = 1$. Thus \mathcal{J}_{ss} is two-dimensional and by Sections 3.1 and 3.2 we have the following possibilities:

1). $\mathcal{J}_{ss} = \mathbb{R}e_1 \oplus \mathbb{R}e_2$. Then $\mathcal{J}^\# = \mathcal{J} \oplus \mathbb{R}1$ contains 3 orthogonal idempotents e_1, e_2 and $e_0 = 1 - e_1 - e_2$, so using the Peirce decomposition (12) we have:

$$\mathcal{J} = \mathcal{P}_{00} \oplus \mathcal{P}_{01} \oplus \mathcal{P}_{02} \oplus \mathcal{P}_{11} \oplus \mathcal{P}_{12} \oplus \mathcal{P}_{22},$$

and the corresponding decomposition of the ideal N :

$$N = N_{00} \oplus N_{01} \oplus N_{02} \oplus N_{11} \oplus N_{12} \oplus N_{22},$$

where $N_{ij} = N \cap \mathcal{P}_{ij}$. Let n_1 be a basis of N , then \mathcal{J} is completely defined by the subspace N_{ij} to which belongs n_1 . Thus \mathcal{J} is one of the following algebras:

\mathcal{J}	Multiplication Table	$\dim \text{Aut}(\mathcal{J})$	$\dim \text{Ann}(\mathcal{J})$	Observation
\mathcal{J}_6	$\mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}n_1$	1	1	associative $n_1 \in N_{00}$
\mathcal{J}_7	$\mathcal{B}_2 \oplus \mathbb{R}e_2$	2	0	$n_1 \in N_{01}$
\mathcal{J}_8	$e_i^2 = e_i \quad e_i n_1 = \frac{1}{2}n_1 \quad i = 1, 2$	2	0	unitary $n_1 \in N_{12}$
\mathcal{J}_9	$\mathcal{B}_1 \oplus \mathbb{R}e_2$	1	0	associative unitary $n_1 \in N_{11}$

Table 3: Three-dimensional Jordan algebras over \mathbb{R} with one-dimensional radical and semi-simple part $\mathcal{J}_{ss} = \mathbb{R}e_1 \oplus \mathbb{R}e_2$.

2). $\mathcal{J}_{ss} = \mathcal{B}_4$. It contains only one idempotent e_1 which determines the following decomposition of N :

$$N = N_0 \oplus N_1 \oplus N_{\frac{1}{2}}.$$

Let n_1 be a basis of N . If $n_1 \in N_0$, then we obtain that $e_2 n_1 \in \mathcal{P}_1 \mathcal{P}_0 = 0$. If $n_1 \in N_1$, we have $e_2 n_1 = \alpha n_1$. Substituting $\{e_1, e_2, n_1\}$ into the Jordan identity (3) we obtain that $\alpha = 0$. Finally there is no real Jordan algebra with $n_1 \in N_{\frac{1}{2}}$.

Thus we conclude that \mathcal{J} is one of the following algebras:

\mathcal{J}	Multiplication Table	$\dim \text{Aut}(\mathcal{J})$	$\dim \text{Ann}(\mathcal{J})$	Observation
\mathcal{J}_{10}	$\mathcal{B}_4 \oplus \mathbb{R}n_1$	1	1	associative $n_1 \in N_0$

\mathcal{J}	Multiplication Table	$\dim \text{Aut}(\mathcal{J})$	$\dim \text{Ann}(\mathcal{J})$	Observation
\mathcal{J}_{11}	$e_2^2 = -e_1 \quad e_1 n_1 = n_1$ $e_1 e_i = e_i \quad i = 1, 2$	2	0	unitary $n_1 \in N_1$

Table 4: Three-dimensional Jordan algebras over \mathbb{R} with one-dimensional radical and semi-simple part $\mathcal{J}_{ss} = \mathcal{B}_4$.

3.3.3. Jordan algebras with two-dimensional radical

Now, suppose that \mathcal{J} is a three-dimensional real Jordan algebra with $\dim N = 2$. The only semi-simple one-dimensional Jordan algebra is $\mathcal{J}_{ss} = \mathbb{R}e_1$, therefore we have the following Peirce decomposition of N :

$$N = N_0 \oplus N_{\frac{1}{2}} \oplus N_1.$$

The ideal N may have two nilpotency types: (2) or (1, 1).

1). Nilpotency type (2). Then $N^2 = 0$. Let $\{n_1, n_2\}$ be a basis of N , then it is enough to choose to which Pierce components belong n_1 and n_2 . Thus \mathcal{J} is one of the following algebras:

\mathcal{J}	Multiplication Table	$\dim \text{Aut}(\mathcal{J})$	$\dim \text{Ann}(\mathcal{J})$	Observation
\mathcal{J}_{12}	$e_1^2 = e_1 \quad e_1 n_i = \frac{1}{2}n_i$ $i = 1, 2$	6	0	$n_1, n_2 \in N_{\frac{1}{2}}$
\mathcal{J}_{13}	$e_1^2 = e_1 \quad e_1 n_i = n_i \quad i = 1, 2$	4	0	associative, unitary $n_1, n_2 \in N_1$
\mathcal{J}_{14}	$\mathcal{B}_2 \oplus \mathbb{R}n_2$	3	1	$n_2 \in N_0, n_1 \in N_{\frac{1}{2}}$
\mathcal{J}_{15}	$\mathcal{B}_1 \oplus \mathbb{R}n_2$	2	1	associative $n_2 \in N_0, n_1 \in N_1$
\mathcal{J}_{16}	$e_1^2 = e_1 \quad e_1 n_1 = \frac{1}{2}n_1$ $e_1 n_2 = n_2$	3	0	$n_1 \in N_{\frac{1}{2}}, n_2 \in N_1$
\mathcal{J}_{17}	$\mathbb{R}e_1 \oplus \mathbb{R}n_1 \oplus \mathbb{R}n_2$	4	2	associative $n_1, n_2 \in N_0$

Table 5: Three-dimensional Jordan algebras over \mathbb{R} with two-dimensional radical of nilpotency type (2).

2). Nilpotency type (1, 1). There exists $n \in N$ such that $N = \mathbb{R}n + \mathbb{R}n^2$ with $n^3 = 0$. Suppose, firstly, that $N = N_i$ then $n^2 \in N_i^2 \subseteq N_0 \oplus N_1$, that implies $i = 0$ or $i = 1$. Now, suppose that $N = N_i \oplus N_{\frac{1}{2}}$ with $i = 0, 1$ and $\dim N_i = \dim N_{\frac{1}{2}} = 1$, we can choose n as an element of $N_{\frac{1}{2}}$. In fact if $N_i = \mathbb{R}a$ and $N_{\frac{1}{2}} = \mathbb{R}b$, then we have $b^2 \in N_i$ thus $b^2 = \alpha a$ for some $\alpha \in \mathbb{R}$. Note that $\alpha \neq 0$ since by nilpotency $a^2 = ab = 0$. Consequently $N = \mathbb{R}b \oplus \mathbb{R}b^2$.

Finally, if $N = N_0 \oplus N_1$, then by nilpotency $N_0^2 = N_1^2 = 0$. Moreover $N_0 N_1 = 0$ and thus $N^2 = 0$ leads to contradiction.

Therefore we obtain that \mathcal{J} is one of the following algebras, where $n_1^2 = n_2$.

\mathcal{J}	Multiplication Table	$\dim \text{Aut}(\mathcal{J})$	$\dim \text{Ann}(\mathcal{J})$	Observation
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\mathcal{J}	Multiplication Table	$\dim \text{Aut}(\mathcal{J})$	$\dim \text{Ann}(\mathcal{J})$	Observation
\mathcal{J}_{18}	$e_1^2 = e_1 \quad n_1^2 = n_2$ $e_1 n_i = n_i \quad i = 1, 2$	2	0	associative, unitary $n_1, n_2 \in N_1$
\mathcal{J}_{19}	$e_1^2 = e_1 \quad n_1^2 = n_2$ $e_1 n_1 = \frac{1}{2} n_1$	2	1	$n_1 \in N_{\frac{1}{2}},$ $n_2 \in N_0$
\mathcal{J}_{20}	$e_1^2 = e_1 \quad n_1^2 = e_1 n_2 =$ n_2 $e_1 n_1 = \frac{1}{2} n_1$	2	0	$n_1 \in N_{\frac{1}{2}},$ $n_2 \in N_1$
\mathcal{J}_{21}	$\mathcal{B}_3 \oplus \mathbb{R} e_1$	2	1	associative $n_1, n_2 \in N_0$

Table 6: Three-dimensional Jordan algebras over \mathbb{R} with two-dimensional radical of nilpotency type $(1, 1)$.

3.3.4. Nilpotent Jordan algebras

We have the following nilpotency types for \mathcal{J} :

1). **Nilpotency type** (3) . Then $\mathcal{J}^2 = 0$.

\mathcal{J}	Multiplication Table	$\dim \text{Aut}(\mathcal{J})$	$\dim \text{Ann}(\mathcal{J})$	Observation
\mathcal{J}_{22}	$\mathbb{R} n_1 \oplus \mathbb{R} n_2 \oplus \mathbb{R} n_3$	9	3	associative

Table 7: Three-dimensional nilpotent real Jordan algebra of nilpotency type (3) .

2). **Nilpotency type** $(1, 1, 1)$. Then $\mathcal{J}^{(4)} = 0$, $\dim \mathcal{J}^3 = 1$ and $\dim \mathcal{J}^2 = 2$. We claim that there exists $n \in \mathcal{J}$ such that $\mathcal{J} = \mathbb{R} n + \mathbb{R} n^2 + \mathbb{R} n^3$. Indeed, let n_1 be a basis for \mathcal{J}^3 , then complete it to a basis of \mathcal{J}^2 and \mathcal{J} , there is $n_2 \in \mathcal{J}^2$ and $n_3 \in \mathcal{J}$, such that $\mathcal{J}^2 = \mathbb{R} n_1 + \mathbb{R} n_2$ and $\mathcal{J} = \mathbb{R} n_1 + \mathbb{R} n_2 + \mathbb{R} n_3$ where

$$n_1^2, n_1 n_2, n_1 n_3 \in \mathcal{J}^{(4)} = 0, \quad n_3^2 = \alpha n_1 + \beta n_2, \text{ since } n_3^2 \in \mathcal{J}^2,$$

$$n_2 n_3 = \gamma n_1 \text{ and } n_2^2 = \delta n_1, \text{ since } n_2 n_3, n_2^2 \in \mathcal{J}^3.$$

Substituting $\{n_1, n_2, n_3\}$ into the Jordan identity (3) we obtain either $\beta = 0$ or $\delta = 0$. But if $\beta = 0$ then $\dim \mathcal{J}^2 = 1$, thus we have $\beta \neq 0$ and $\delta = 0$, analogously $\gamma \neq 0$. Then $n = n_3$ and $\{n, n^2, n^3\}$ is the desired basis. Thus we obtain the following algebra:

\mathcal{J}	Multiplication Table	$\dim \text{Aut}(\mathcal{J})$	$\dim \text{Ann}(\mathcal{J})$	Observation
\mathcal{J}_{23}	$n_2 n_3 = n_1 \quad n_3^2 = n_2$	3	1	associative

Table 8: Three-dimensional nilpotent real Jordan algebra of nilpotency type $(1, 1, 1)$.

3). **Nilpotency type** $(2, 1)$. We observe that $\mathcal{J}^3 = 0$ and $\dim \mathcal{J}^2 = 1$. Let n_3 be a basis of \mathcal{J}^2 , complete it to a basis of \mathcal{J} choosing some $n_1, n_2 \in \mathcal{J}$, we have $\mathcal{J} = \mathbb{R} n_1 + \mathbb{R} n_2 + \mathbb{R} n_3$

with

$$\begin{aligned} n_3^2, n_1 n_3, n_2 n_3 &\in \mathcal{J}^3 = 0, \\ n_1^2 &= \alpha n_3, n_2^2 = \beta n_3, n_1 n_2 = \gamma n_3. \end{aligned} \tag{22}$$

If (\mathcal{J}, \cdot) is an algebra with basis $\{N_1, N_2, N_3\}$ whose multiplication \cdot satisfies (22) then for any $\alpha, \beta, \gamma \in \mathbb{R}$, \mathcal{J} is a Jordan algebra, but at least one of them has to be not null. We claim that \mathcal{J} is one of the following algebras:

\mathcal{J}	Multiplication Table	$\dim \text{Aut}(\mathcal{J})$	$\dim \text{Ann}(\mathcal{J})$	Observation
\mathcal{J}_{24}	$n_1^2 = n_2^2 = n_3$	4	1	associative
\mathcal{J}_{25}	$\mathcal{B}_3 \oplus \mathbb{R}n_3$	5	2	associative
\mathcal{J}_{26}	$n_1 n_2 = n_3$	4	1	associative

Table 9: Three-dimensional nilpotent real Jordan algebras of nilpotency type $(2, 1)$.

Indeed, let $\beta = 0$ then, if $\gamma \neq 0$ we obtain that $\mathcal{J} \simeq \mathcal{J}_{26}$ where the isomorphism is given by $N_1 \mapsto n_1 + \frac{\alpha}{2\gamma}n_2$, $N_2 \mapsto n_2$ and $N_3 \mapsto \gamma^{-1}n_3$. If $\gamma = 0$, necessarily $\alpha \neq 0$ and $\mathcal{J} \simeq \mathcal{J}_{25}$ with change of basis: $N_1 \mapsto n_1$, $N_2 \mapsto n_3$ and $N_3 \mapsto \alpha^{-1}n_2$.

On the other hand, denote $-\alpha\beta + \gamma^2$ by Δ and suppose $\beta \neq 0$, then:

- i If $\Delta > 0$, again $\mathcal{J} \simeq \mathcal{J}_{26}$ via $N_1 \mapsto \left(\frac{1}{2} + \frac{\gamma}{2\sqrt{\Delta}}\right)n_1 + \left(-\frac{1}{2} + \frac{\gamma}{2\sqrt{\Delta}}\right)n_2$, $N_2 \mapsto \frac{\beta}{2\sqrt{\Delta}}n_1 + \frac{\beta}{2\sqrt{\Delta}}n_2$ and $N_3 \mapsto \frac{\beta}{2\Delta}n_3$.
- ii If $\Delta < 0$, in this case $\mathcal{J} \simeq \mathcal{J}_{24}$ via $N_1 \mapsto n_1 + \frac{\gamma}{\sqrt{-\Delta}}n_2$, $N_2 \mapsto \frac{\beta}{\sqrt{-\Delta}}n_2$ and $N_3 \mapsto -\frac{\beta}{\Delta}n_3$.
- iii If $\Delta = 0$ then $\mathcal{J} \simeq \mathcal{J}_{25}$ in both cases: if $\gamma = 0$ with isomorphism $N_1 \mapsto n_3$, $N_2 \mapsto n_1$ and $N_3 \mapsto \beta^{-1}n_2$, and if $\gamma \neq 0$ with isomorphism given by $N_1 \mapsto n_1 + n_3$, $N_2 \mapsto \alpha^{-1}\gamma n_1$ and $N_3 \mapsto \alpha^{-1}n_2$.

3.4. Remarks

We prove that given \mathcal{J} a three-dimensional Jordan algebra over \mathbb{R} then \mathcal{J} is one of the algebras \mathcal{J}_1 to \mathcal{J}_{26} . To complete the algebraic classification just remains to prove that all them are pairwise non-isomorphic. Comparing the algebra invariants, namely $\dim \text{Rad}(\mathcal{J})$, $\dim \text{Ann}(\mathcal{J})$, $\dim \text{Aut}(\mathcal{J})$, nilpotency type of $\text{Rad}(\mathcal{J})$, together with properties whether \mathcal{J} is indecomposable, associative, non-associative, unitary, one needs only to verify whether there exist isomorphisms between \mathcal{J}_3 , \mathcal{J}_4 and \mathcal{J}_5 and between \mathcal{J}_{24} and \mathcal{J}_{26} . First, we observe that the algebras \mathcal{J}_3 , \mathcal{J}_4 and \mathcal{J}_5 are associated to non-degenerate symmetric bilinear forms that are non-isomorphic and therefore they are pairwise non-isomorphic. Finally, the two-dimensional null algebra $\mathbb{R}n_1 \oplus \mathbb{R}n_2$ is a subalgebra of \mathcal{J}_{26} but is not a subalgebra of \mathcal{J}_{24} , thus $\mathcal{J}_{24} \not\simeq \mathcal{J}_{26}$.

4 Geometric Classification

In this section we will determine the geometric classification of three-dimensional real Jordan algebras. But, due to item (v) of Proposition 21 which gives a sufficient condition of existence of deformation between decomposable algebras, first we will describe the varieties of Jordan algebras of dimension less than three. The only rigid one-dimensional real Jordan

algebra is the simple one $\mathbb{R}e$ and it is clear that $\mathbb{R}e \rightarrow \mathbb{R}n$. Therefore, $\mathcal{J}_{\text{or}_1}$ is an irreducible algebraic variety of dimension 1 with two G-orbits. In [1] the authors proved that $\mathcal{J}_{\text{or}_2}$ is an algebraic variety with 7 orbits under the action of G and 3 irreducible components given by the Zariski closure of the orbits of the algebras $\mathbb{R}e_1 \oplus \mathbb{R}e_2$, \mathcal{B}_2 and \mathcal{B}_4 . The deformations between the algebras in $\mathcal{J}_{\text{or}_2}$ are represented in Figure 1.

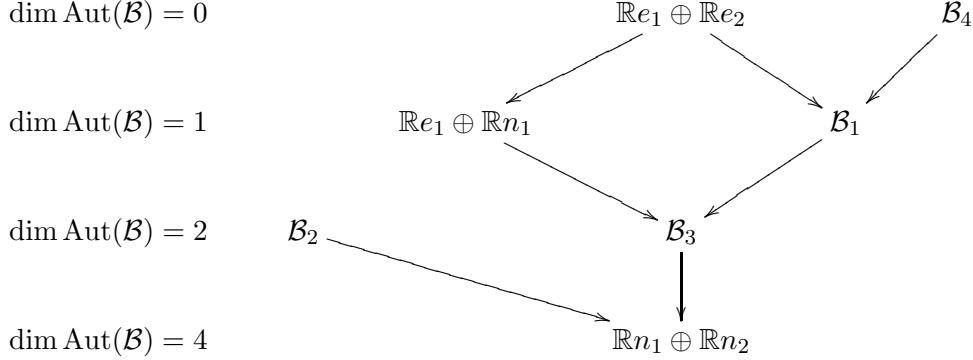


Figure 1: Complete description of the G-orbits of $\mathcal{J}_{\text{or}_2}$

Now, we are ready to determine the irreducible components of the variety of three dimensional real Jordan algebras.

Theorem 23. *The variety $\mathcal{J}_{\text{or}_3}$ of three-dimensional real Jordan algebras is a connected affine variety of dimension 9 with 26 orbits under the action of $\text{GL}(V)$ and 8 irreducible components given by Zariski closure of the orbits of the following algebras:*

$$\Omega = \{\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_4, \mathcal{J}_5, \mathcal{J}_7, \mathcal{J}_{12}, \mathcal{J}_{20}\}.$$

Proof. For any $\mathcal{J} \in \mathcal{J}_{\text{or}_3}$, the change of basis

$$g(t) = \begin{bmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{bmatrix},$$

gives $\mathcal{J} \rightarrow \mathcal{J}_{22}$ then the orbit \mathcal{J}_{22}^G belongs to any irreducible component of $\mathcal{J}_{\text{or}_3}$ and hence $\mathcal{J}_{\text{or}_3}$ is a connected affine variety. The fact that it has 26 G-orbits follows of the algebraic classification in Section 3. Then, $\mathcal{J}_{\text{or}_3} = \bigcup_{i=1}^{26} \mathcal{J}_i^G$ is a finite union of orbits which are locally closed sets, thus

$$\dim \mathcal{J}_{\text{or}_3} = \max_{1 \leq i \leq 26} \left\{ \dim \mathcal{J}_i^G \right\} = \dim \mathcal{J}_1^G = 3^2 - \dim \text{Aut}(\mathcal{J}_1) = 9.$$

We will divide the rest of the proof in two parts. Firstly, we will show that all algebras in Ω are rigid and then, we will prove that there is no other rigid algebra in $\mathcal{J}_{\text{or}_3}$, that is, every structure from \mathcal{J}_1 to \mathcal{J}_{26} in the algebraic classification is dominated by one of the algebras from Ω .

The algebras \mathcal{J}_1 and \mathcal{J}_2 have $\dim \text{Aut}(\mathcal{J}_i) = 0$ then, by Proposition 21(i), no other algebra in $\mathcal{J}_{\text{or}_3}$ can be a deformation of them. Therefore \mathcal{J}_1 and \mathcal{J}_2 are rigid algebras. By the same argument the only algebras that could be a deformation of \mathcal{J}_3 , \mathcal{J}_4 and \mathcal{J}_5 are \mathcal{J}_1 and \mathcal{J}_2 but, by Proposition 21(vi), an associative algebra could not be a deformation of a non-associative one. Thus \mathcal{J}_3 , \mathcal{J}_4 and \mathcal{J}_5 also are rigid algebras.

There is no algebra in \mathcal{Jor}_3 which dominates \mathcal{J}_7 : by Proposition 21(ii) the only possible candidates to be deformations of \mathcal{J}_7 are those whose $\dim \text{Rad}(\mathcal{J}_i) \leq 1$, that is, the algebras \mathcal{J}_1 to \mathcal{J}_{11} . By Proposition 21(vi), we may exclude $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_6, \mathcal{J}_9$ and \mathcal{J}_{10} from the list. Since $\dim \text{Aut}(\mathcal{J}_i) = 2$, for $i = 7, 8, 11$ neither \mathcal{J}_8 or \mathcal{J}_{11} could dominate \mathcal{J}_7 due to Proposition 21(i). The dimension of the 2-cocycle groups of the three remainder algebras, \mathcal{J}_i with $i = 3, 4, 5$, is 8 while $\dim Z^2(\mathcal{J}_7, \mathcal{J}_7) = 7$, thus by Proposition 21(vii) none of them dominates \mathcal{J}_7 . We conclude that \mathcal{J}_7 is rigid.

It follows from Example 19 that the algebra \mathcal{J}_{12} is rigid.

For the proof of the rigidness of \mathcal{J}_{20} we use the same arguments as for \mathcal{J}_7 : Since $\dim \text{Rad}(\mathcal{J}_{20}) = 2$ no nilpotent algebra, i.e. \mathcal{J}_i for $22 \leq i \leq 26$, is a deformation of \mathcal{J}_{20} . By the argument of dimension of the automorphism group, the algebras \mathcal{J}_i for $i = 7, 8, 11, \dots, 21$ do not dominate \mathcal{J}_{20} . Also we can exclude from the list the associative algebras $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_6, \mathcal{J}_9$ and \mathcal{J}_{10} because \mathcal{J}_{20} is non-associative. Finally, $\dim Z^2(\mathcal{J}_{20}, \mathcal{J}_{20}) = 7$ implies that \mathcal{J}_i for $i = 3, 4, 5$ are not deformations of \mathcal{J}_{20} and therefore the algebra \mathcal{J}_{20} is rigid.

It remains to show that for any algebra \mathcal{J}_i for $i = 1, \dots, 26$ there is an algebra $\mathcal{J} \in \Omega$ such that \mathcal{J} is a deformation of \mathcal{J}_i . In what follows all transformations are given using the basis from Section 3.

Firstly, the orbits of the algebras $\mathcal{J}_6, \mathcal{J}_{21}$ and \mathcal{J}_{24} , are contained in $\overline{\mathcal{J}_1^G}$: combining the deformation between two-dimensional real Jordan algebras obtained in [1] with Proposition 21(v) we have: $\mathcal{J}_1 \rightarrow \mathcal{J}_6$ and $\mathcal{J}_6 \rightarrow \mathcal{J}_{21}$. If we consider the family of automorphisms of \mathcal{J}_{21} given by $A_t = tn_1, B_t = t^2e_1 - n_2$ and $C_t = t^2n_2$, making t tends to 0 we obtain the algebra \mathcal{J}_{24} , thus $\mathcal{J}_{21} \rightarrow \mathcal{J}_{24}$.

The orbits of the $\mathcal{J}_9, \mathcal{J}_{10}, \mathcal{J}_{13}, \mathcal{J}_{15}, \mathcal{J}_{18}, \mathcal{J}_{23}, \mathcal{J}_{25}$, and \mathcal{J}_{26} , belong to $\overline{\mathcal{J}_2^G}$: again, combining the results for dimension two in [1] with Proposition 21(v) we obtain: $\mathcal{J}_2 \rightarrow \mathcal{J}_9, \mathcal{J}_2 \rightarrow \mathcal{J}_{10}$ and $\mathcal{J}_{10} \rightarrow \mathcal{J}_{15}$. From [9] we get the following deformations over \mathbb{R} : $\mathcal{J}_9 \rightarrow \mathcal{J}_{18}, \mathcal{J}_{18} \rightarrow \mathcal{J}_{13}, \mathcal{J}_{23} \rightarrow \mathcal{J}_{26}$ and $\mathcal{J}_{26} \rightarrow \mathcal{J}_{25}$. To see that $\mathcal{J}_{15} \rightarrow \mathcal{J}_{23}$ take, for $t \neq 0$, the change of basis $A_t = t^2n_2, B_t = tn_1 - tn_2$ and $C_t = te_1 + n_1 + n_2$ of \mathcal{J}_{15} . Since $B_tC_t = A_t + tB_t$ and $C_t^2 = B_t + tC_t$ we get the structure of \mathcal{J}_{23} when t tends to zero.

The rigid algebra \mathcal{J}_3 dominates \mathcal{J}_8 and \mathcal{J}_{14} : consider, for $t \neq 0$, the family of automorphisms of \mathcal{J}_3 given by $A_t = \frac{1}{2}e_1 + \frac{1}{2}e_2, B_t = \frac{1}{2}e_1 - \frac{1}{2}e_2$ and $C_t = te_3$ when we make t tends to 0 we obtain \mathcal{J}_8 , thus $\mathcal{J}_3 \rightarrow \mathcal{J}_8$. From [9] we get $\mathcal{J}_8 \rightarrow \mathcal{J}_{14}$.

The algebra \mathcal{J}_{11} belong to the Zariski closure of the orbit of \mathcal{J}_4 . To show that consider, for $t \neq 0$, the change of basis of \mathcal{J}_4 : $A_t = e_1, B_t = e_2$ and $C_t = te_3$ when t tends to 0 we obtain the algebra \mathcal{J}_{11} . From [9] and [1] it follows that $\mathcal{J}_{20} \rightarrow \mathcal{J}_{16}$ and $\mathcal{J}_7 \rightarrow \mathcal{J}_{17}$, respectively.

Lastly, it remains to show that the rigid algebra \mathcal{J}_5 dominates \mathcal{J}_{19} . For $t \neq 0$, it is sufficient to consider the change of basis: $A_t = \frac{1}{2}e_1 - \frac{1}{2}e_2, B_t = te_3$ and $C_t = \frac{t^2}{2}e_1 + \frac{t^2}{2}e_2$ of \mathcal{J}_5 . Since $B_t^2 = C_t + t^2A_t, B_tC_t = \frac{t^2}{2}B_t$ and $C_t^2 = t^2C_t$ the structural constants of \mathcal{J}_5 tends to those of \mathcal{J}_{19} when t tends to zero. This completes the geometric description of the variety \mathcal{Jor}_3 . The orbits of \mathcal{Jor}_3 are represented in Figure 2. \square

$\dim \text{Aut}(\mathcal{J})$

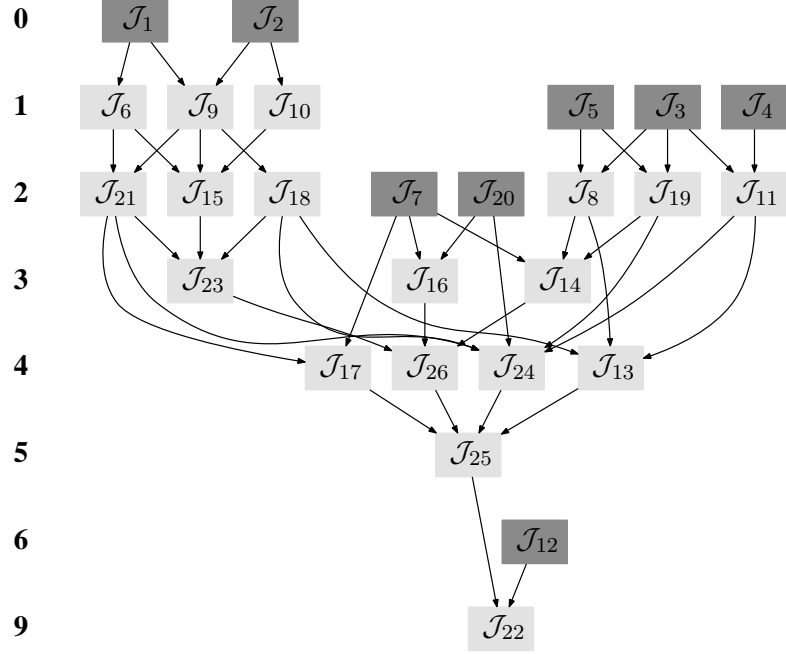


Figure 2: Description of the G-orbits of the variety $\mathcal{J}_{\text{or}_3}$

4.1. Final Remarks

We do not provide the complete list of deformations between algebras in $\mathcal{J}_{\text{or}_3}$. For any non-rigid algebra \mathcal{J} we found at least one rigid which dominates \mathcal{J} , for further examples of deformation in $\mathcal{J}_{\text{or}_3}$ see [11].

We note that any algebra \mathcal{J} in Ω satisfies the sufficient condition for rigidity of Proposition 18, namely $H^2(\mathcal{J}, \mathcal{J}) = 0$.

The informations known until the moment about the number of orbits and irreducible components of the variety $\mathcal{J}_{\text{or}_n}$ when the base field is \mathbb{R} or \mathbb{C} are reunited in Table 10.

	$\mathcal{J}_{\text{or}_n}^{\mathbb{R}}$		$\mathcal{J}_{\text{or}_n}^{\mathbb{C}}$	
n	No. orbits	No. components	No. orbits	No. components
1	2	1	2	1
2	7	3	6	2
3	26	8	20	5
4	> 109	≥ 18	73	10
5	-	-	$\gg 223$	≥ 26

Table 10: Comparison of varieties $\mathcal{J}_{\text{or}_n}^{\mathbb{C}}$ and $\mathcal{J}_{\text{or}_n}^{\mathbb{R}}$

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